Decidability of Modal System S5: A Recursive Algorithm.

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Abstract.- The decidability method, given in [6], for modal system S5 uses the reduced modal normal form. In this paper we present a recursive algorithm for computing the reduced modal normal form and use this algorithm as a subroutine for an algorithm deciding validity of formulas of system S5.

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0.- Introduction

Modal logics ([6]) have found a variety of uses in Artificial Intelligence and in Computer Science (cfr. [8], [2], [9]). For such applications, efficient automated proof systems are very desirable. There are some decision procedures for modal logics (e.g. [1], [3], [4], [5], [7]). In this paper, we present a recursive algorithm for deciding validity of formulas of system S5.

1.- Notations and terminology

We shall denote by X_1, X_2,.... the propositional variables, and for primitive connectives, we shall use the symbols $\neg$ (negation), $\vee$ (disjunction), $\Box$ (necessarity).

Definition 1.1. We define the set of formulas **P** inductively as follows:
(a) The propositional variables are formulas.
(b) If P is a formula, then $\neg P$ and $\Box P$ are formulas.
(c) If P and Q are formulas, then $P \vee Q$ is a formula.
We shall use capital letters P, Q, R,... for metavariables about the formulas.

Definition 1.2. We define new connectives writting:

$$
\begin{array}{lll}
P \wedge Q & = \neg(\neg P \vee \neg Q) & (P \text{ and } Q) \\
P \rightarrow Q & = \neg P \vee Q & (\text{if } P, \text{ then } Q) \\
P \leftrightarrow Q & = (P \rightarrow Q) \wedge (Q \rightarrow P) & (P \text{ iff } Q) \\
\Diamond P & = \neg(\Box \neg P) & (P \text{ is possible})
\end{array}
$$

Definition 1.3. The set of **axioms** in S5 are:
AX1: $P \vee P \rightarrow P$
AX2: $P \rightarrow P \vee Q$
AX3: $P \vee Q \rightarrow Q \vee P$
AX4: $(Q \rightarrow R) \rightarrow (P \vee Q \rightarrow P \vee R)$
AX5: $\Box P \rightarrow P$
AX6: $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$
AX7: $\Diamond P \rightarrow \Box \Diamond P$

Definition 1.4. The **theorems** in S5 are defined inductively as follows:
(a) The axioms are theorems.
(b) If $P \rightarrow Q$ and P are theorems, then Q is a theorem (**Modus-Ponens Rule**).
(c) If P is a theorem, $\Box P$ is a theorem (**Necessarity Rule**).
If P is a theorem in S5, we put $\vdash P$.

Theorem 1.5. If P is a tautology in classical propositional calculus, then $\vdash P$.

Corollary 1.6. The following formulas are theorems in S5:
(1) $\neg(\neg Q) \leftrightarrow Q$
(2) $\neg(Q \wedge R) \leftrightarrow \neg Q \vee \neg R$
(3) $\neg(Q \vee R) \leftrightarrow \neg Q \wedge \neg R$
(4) $(Q \wedge R) \vee S \leftrightarrow (Q \vee S) \wedge (R \vee S)$
(5) $Q \vee (R \wedge S) \leftrightarrow (Q \vee R) \wedge (Q \vee S)$
(6) $(Q \vee R) \wedge S \leftrightarrow (Q \wedge S) \vee (R \wedge S)$
(7) $Q \wedge (R \vee S) \leftrightarrow (Q \wedge R) \vee (Q \wedge S)$

Theorem 1.7. If $\vdash P_1, \ldots, \vdash P_m$ and Q is a tautological consequence of $P_1, \ldots, P_m$, then $\vdash Q$.

Theorem 1.8. The following formulas are theorems in S5:
(1) $\neg(\Diamond P) \leftrightarrow \Box(\neg P)$
(2) $\neg(\Box P) \leftrightarrow \Diamond(\neg P)$
(3) $\Box(\Box P) \leftrightarrow \Box P$
(4) $\Box(\Diamond P) \leftrightarrow \Diamond P$
(5) $\Box(P \wedge Q) \leftrightarrow (\Box P) \wedge (\Box Q)$
(6) $\Box((\Box P) \vee Q) \leftrightarrow \Box(\Box P) \vee (\Box Q)$
(7) $\Box((\Diamond P) \vee Q) \leftrightarrow \Box(\Diamond P) \vee (\Box Q)$
(8) $\Diamond(\Box P) \leftrightarrow \Box P$
(9) $\Diamond(\Diamond P) \leftrightarrow \Diamond P$
(10) $\Diamond(P \vee Q) \leftrightarrow (\Diamond P) \vee (\Diamond Q)$
(11) $\Diamond((\Box P) \wedge Q) \leftrightarrow \Diamond(\Box P) \wedge (\Diamond Q)$
(12) $\Diamond(P \vee Q) \leftrightarrow (\Diamond P) \vee (\Diamond Q)$
(13) $\Diamond((\Box P) \wedge Q) \leftrightarrow \Diamond(\Box P) \wedge (\Diamond Q)$
(14) $\Diamond((\Diamond P) \wedge Q) \leftrightarrow \Diamond(\Diamond P) \wedge (\Diamond Q)$
(15) $\Box(P \vee \Diamond Q) \rightarrow \Box P \vee \Diamond Q$
(16) $P \rightarrow (\Diamond P)$

The proofs of these theorems are in [6].

2.-Extended normal forms

Definition 2.1.

(a) A **literal** is either a propositional variable X_i, or the negation $\neg X_i$ of an propositional variable.

(b) A formula P is an **extended clause** if P is a disjunction where each disjunct is either a literal, or $\Box Q$, or $\Diamond Q$.

(c) A formula P is in **extended conjunctive normal form** (**E.C.N.F.**) if it is of the form:

$$C_1 \wedge \cdots \wedge C_m,$$

where $m \geq 1$ and each C_i is an extended clause.

(d) A formula P is an **extended cube** if P is a conjunction where each conjunct is either a literal, or $\Box Q$, or $\Diamond Q$.

(c) A formula P is in **extended disjunctive normal form** (**E.D.N.F.**) if it is of the form:

$$C_1 \vee \cdots \vee C_m,$$

where $m \geq 1$ and each C_i is an extended cube.

Definition 2.2. Let $f_1 : \mathbf{P} \to \mathbf{P}$ be the map defined inductively as follows:

$$f_1(P) = \begin{cases} P, & \text{if } P \text{ is a propositional variable;} \\ \neg f_1(Q) \vee f_1(R), & \text{if } P \text{ is } Q \to R; \\ f_1(Q \to R) \wedge f_1(R \to Q), & \text{if } P \text{ is } Q \leftrightarrow R; \\ k f_1(Q), & \text{if } P \text{ is } kQ, \text{ where } k \text{ is } \neg, \Box \text{ or } \Diamond; \\ f_1(Q) k f_1(R), & \text{if } P \text{ is } QkR, \text{ where } k \text{ is } \vee \text{ or } \wedge. \end{cases}$$

Let us write $\mathbf{P_1} = f_1(\mathbf{P})$.

Theorem 2.3. Let P be an element of $\mathbf{P}$. Then:

(a) $\vdash P \leftrightarrow f_1(P)$.

(b) In $f_1(P)$ the connectives $\to$, $\leftrightarrow$ do not appear.

Proof: Is a consequence of 1.2. ∎

Definition 2.4. Let $f_2 : \mathbf{P}_1 \to \mathbf{P}$ be the map defined inductively as follows:

$$f_2(P) = \begin{cases} P, & \text{if } P \text{ is a literal;} \\ f_2(Q), & \text{if } P \text{ is } \neg(\neg Q); \\ f_2(\neg Q) \vee f_2(\neg R), & \text{if } P \text{ is } \neg(Q \wedge R); \\ f_2(\neg Q) \wedge f_2(\neg R), & \text{if } P \text{ is } \neg(Q \vee R); \\ \Box f_2(\neg Q), & \text{if } P \text{ is } \neg(\Diamond Q); \\ \Diamond f_2(\neg Q), & \text{if } P \text{ is } \neg(\Box Q); \\ k f_2(Q), & \text{if } P \text{ is } kQ, \text{ where } k \text{ is } \Box \text{ or } \Diamond; \\ f_2(Q) k f_2(R), & \text{if } P \text{ is } QkR, \text{ where } k \text{ is } \wedge \text{ or } \vee. \end{cases}$$

Let us put $\mathbf{P}_2 = f_2(\mathbf{P}_1)$.

Theorem 2.5. Let P be an element of $\mathbf{P}_1$. Then:
(a) $\vdash P \leftrightarrow f_2(P)$.
(b) In $f_2(P)$ the connectives $\rightarrow$, $\leftrightarrow$ do not appear.
(c) In $f_2(P)$, $\neg$ appears only immediately before propositional variables.

Proof: Is a consequence of 1.6.(1) – (3) and 1.8.(1) – (2). ∎

Definition 2.6. Let $f_3 : \mathbf{P}_2 \rightarrow \mathbf{P}$ be the map defined as follows:
(1) If P is in ECNF, then $f_3(P) = P$.
(2) If P is not in ECNF, then:
(2.1) if P is $Q \wedge R$, then $f_3(P) = f_3(Q) \wedge f_3(R)$;
(2.2) if P is $(Q \wedge R) \vee S$, then $f_3(P) = f_3(Q \vee S) \wedge f_3(R \vee S)$;
(2.3) if P is $Q \vee (R \wedge S)$, then $f_3(P) = f_3(Q \vee R) \wedge f_3(Q \vee S)$;
(2.4) Otherwise, P is $Q \vee R$, then $f_3(P) = f_3(Q) \vee f_3(R)$.

Theorem 2.7. Let P be an element of $\mathbf{P}_2$. Then:
(a) $\vdash P \leftrightarrow f_3(P)$.
(b) $f_3(P)$ is in ECNF.

Proof: Is a consequence of 1.6.(4) – (5). ∎

Corollary 2.8. Let P be a formula of $\mathbf{P}$. Then
(a) $f_3(f_2(f_1(P))))$ is in ECNF.
(b) $\vdash P \leftrightarrow f_3(f_2(f_1(P))))$.

Definition 2.9. Let $f_4 : \mathbf{P}_2 \rightarrow \mathbf{P}$ be the map defined inductively as follows:
(1) If P is in EDNF, then $f_4(P) = P$.
(2) If P is not in EDNF, then:
(2.1) if P is $Q \vee R$, then $f_4(P) = f_4(Q) \vee f_4(R)$;
(2.2) if P is $(Q \vee R) \wedge S$, then $f_4(P) = f_4(Q \wedge S) \vee f_4(R \wedge S)$;
(2.3) if P is $Q \wedge (R \vee S)$, then $f_4(P) = f_4(Q \wedge R) \vee f_4(Q \wedge S)$;
(2.4) Otherwise, P is $Q \wedge R$, then $f_4(P) = f_4(Q) \wedge f_4(R)$.

Theorem 2.10. Let P be an element of $\mathbf{P}_2$. Then:
(a) $\vdash P \leftrightarrow f_4(P)$.
(b) $f_4(P)$ is in EDNF.

Proof: Is a consequence of 1.6.(6) – (7). ∎

Corollary 2.11. Let P be a formula of $\mathbf{P}$. Then
(a) $f_4(f_2(f_1(P)))$ is in EDNF.
(b) $\vdash P \leftrightarrow f_4(f_2(f_1(P)))$

3.- Normal forms

Definition 3.1. We define the **degree** of a formula P, $\deg(P)$, inductively as follows:

$$\deg(P) = \begin{cases} 0, & \text{if } P \text{ is a propositional variable;} \\ \deg(Q), & \text{if } P \text{ is } \neg Q; \\ 1+\deg(Q), & \text{if } P \text{ is } kQ, \text{ where } k \text{ is } \Box \text{ or } \Diamond; \\ \max(\deg(Q), \deg(R)), & \text{if } P \text{ is } QkR, \text{ where } k \text{ is } \vee, \wedge, \rightarrow \text{ or } \leftrightarrow. \end{cases}$$

Definition 3.2.

(a) A formula P is a **clause** if P is a disjunction where each disjunct is either a literal, or $\Box Q$, or $\Diamond Q$, where $\deg(Q) = 0$.

(b) A clause P is **basic** if $\deg(P) = 0$.

(c) A formula P is in **conjuctive normal form** (**C.N.F.**) if P is a conjunction of clauses.

Definition 3.3. Let $f_5 : \mathbf{P}_2 \to \mathbf{P}$ be the map defined inductively as follows:

(1) If $\deg(P) \leq 1$, then $f_5(P) = P$.

(2) If $\deg(P) > 1$, then:

(2.1) if P is $Q \wedge R$, then $f_5(Q) \wedge f_5(R)$;

(2.2) if P is $Q \vee R$, then $f_5(P) = f_5(Q) \vee f_5(R)$;

(2.3) if P is $\Box(Q \wedge R)$, then $f_5(P) = f_5(\Box Q) \wedge f_5(\Box R)$;

(2.4) if P is $\Box(Q \vee R)$, then:

(2.4.1) if $Q \vee R$ is in ECNF, then P is

$$\Box(P_1 \wedge \cdots \wedge P_n)$$

and

$$f_5(P) = f_5(P_{i_1} \wedge \cdots \wedge P_{i_k}) \wedge \Box(P_{i_{k+1}} \wedge \cdots \wedge P_{i_n})$$

where $P_{i_1}, \ldots, P_{i_k}$ $(1 \leq i_1 < \cdots < i_k \leq n)$ are the P_i that begins with $\Box$ or $\Diamond$, and $P_{i_{k+1}}, \ldots, P_{i_n}$ $(1 \leq i_{k+1} < \ldots < i_n \leq n)$ are the other ones;

(2.4.2) if $Q \vee R$ is not in EDNF, then $f_5(P) = f_5(\Box f_3(P))$;

(2.5) if P is $\Box(kQ)$, where k is $\Box$ or $\Diamond$, then $f_5(P) = f_5(kQ)$;

(2.6) if P is $\Diamond(Q \vee R)$, then $f_5(P) = f_5(\Diamond Q) \vee f_5(\Diamond R)$;

(2.7) if P is $\Diamond(Q \wedge R)$, then:

(2.7.1) if $Q \wedge R$ is in EDNF, then P is

$$\Diamond(P_1 \vee \cdots \vee P_m)$$

and

$$f_5(P) = f_5(P_{i_1} \vee \cdots \vee P_{i_k}) \vee P_{i_{k+1}} \vee \cdots \vee P_{i_n}$$

where $P_{i_1}, \ldots, P_{i_k}$ $(1 \leq i_1 < \ldots < i_k \leq n)$ are the P_i that begins with $\Box$ or $\Diamond$, and $P_{i_{k+1}}, \ldots, P_{i_n}$ $(1 \leq i_{k+1} < \cdots < i_n \leq n)$ are the other ones;

(2.7.2) if $Q \wedge R$ is not in ECNF, then $f_5(P) = f_5(\Diamond f_4(P))$;

(2.8) if P is $\Diamond(kQ)$, where k is $\Diamond$ or $\Box$, then $f_5(P) = f_5(kQ)$;

Theorem 3.4. Let P be an element of $\mathbf{P}_2$, then

(a) $\vdash P \leftrightarrow f_5(P)$.

(b) $\deg(f_5(P)) \leq 1$.

Proof: Is a consequence of 1.8.(3) – (14). ∎

Theorem 3.5. Let P be a formula of $\mathbf{P}_2$. Then
(a) $\deg(f_5(f_2(f_1(P)))) \leq 1$.
(b) $\vdash P \leftrightarrow f_5(f_2(f_1(P)))$.

Theorem 3.6. Let P be a formula of $\mathbf{P}$. Then
(a) $f_3(f_5(f_2(f_1(P))))$ is in CNF.
(b) $\vdash P \leftrightarrow f_3(f_5(f_2(f_1(P))))$.

4.- The vadility in S5

Definition 4.1.
(a) A **model** $\mathbf{M}$ of S5 is a pair (M, V), where M is a non-empty set (of 'possible worlds') and

$$V : \{X_1, X_2, \dots\} \times M \to \{0, 1\}.$$

(b) Let $(\mathbf{M}, V)$ a model of S5. We define the map $V : \mathbf{P} \times M \to \{0, 1\}$ inductively as follows:
(b.1) $V(X_i, m) = V(X_i, m)$.
(b.2) $V(\neg P, m) = 1$ iff $V(P, m) = 0$.
(b.3) $V(P \vee Q, m) = 0$, iff $V(P, m) = V(Q, m) = 0$.
(b.4) $V(\Box P, m) = 1$ iff $V(P, m') = 1$, for every $m' \in M$.

Lemma 4.2. Let (M, V) be a model of S5, and let P, Q be elements of $\mathbf{P}$, $m \in M$. Then:
(a) $V(P \wedge Q, m) = 1$, iff $V(P, m) = V(Q, m) = 1$.
(b) $V(P \to Q, m) = 0$, iff $V(P, m) = 1$ and $V(Q, m) = 0$.
(c) $V(P \leftrightarrow Q, m) = 1$, iff $V(P, m) = V(Q, m)$.
(d) $V(\Diamond P, m) = 1$, iff there exists an element $m' \in M$ such that $V(P, m') = 1$.

Definition 4.3.
(a) A formula P is **valid in** a model $\mathbf{M} = (M, V)$ of S5, $\mathbf{M} \models P$, if $V(P, m) = 1$, for every $m \in M$.
(b) P is **valid,** $\models P$, if P is valid in any model $\mathbf{M}$ of S5.

Example 4.4. The formula $X \to \Box X$ is not valid: let us consider $M = \{m_1, m_2\}$ and the map V such that $V(X, m_1) = 1$ and $V(X, m_2) = 0$. Then $V(\Box X, m_1) = 0$ and $V(X \to \Box X, m_1) = 0$, so $X \to \Box X$ is not valid.

Theorem 4.5. $\vdash P \iff \models P$

Proof: In [6]. ∎

5.- Description of the algorithm

Lemma 5.1. Let $P_1, \dots, P_k$ be elements of $\mathbf{P}$. The following conditions are equivalent:

(a) $\vdash P_1 \wedge \cdots \wedge P_k$.
(b) $\vdash P_1, \ldots, \vdash P_k$.

Proof: Is a consequence of 4.5 and 4.3.

Lemma 5.2. Let C be a clause. Then there exist a clause C' such that:
(a) C' has the following form:

$$P \vee \Box Q_1 \vee \cdots \vee \Box Q_n \vee \Diamond R,$$

where $\deg(P) = \deg(Q_1) = \cdots = \deg(Q_n) = \deg(R) = 0$.
(b) $\vdash C \leftrightarrow C'$.

Proof: It is clear using that the disjunction is associative and commutative, and 1.8(10). ∎

Theorem 5.3. Let C be the formula

$$P \vee \Box Q_1 \vee \cdots \vee \Box Q_k \vee \Diamond R$$

where $\deg(P) = \deg(Q_1) = \cdots = \deg(Q_n) = \deg(R) = 0$. The following conditions are equivalent:
(a) C is a theorem in S5.
(b) Some of the formulas $P \vee R, Q_1 \vee R, \ldots, Q_k \vee R$ is a theorem in the classical propositional calculus.

Proof:

(a) $\Longrightarrow$ (b) Let us assume that the formulas $P \vee R, Q_1 \vee R, \ldots, Q_k \vee R$ are not theorems in the propositional calculus. Then, there exists valuations $v_0, v_1, \ldots, v_k$ such that

$$v_0(P \vee R) = v_1(Q_1 \vee R) = \cdots = v_k(Q_k \vee R) = 0.$$

Let us consider the following model (M, V):
$M = \{m_0, m_1, \ldots, m_k\}$,
$V : \mathbf{P} \times M \to \{0, 1\}$ is the map defined by:

$$V(X_i, m_j) = v_j(X_i),\ 1 \leq i,\ 0 \leq j \leq k.$$

It is clear that $V(C, m_0) = 0$, so C is not valid, and, by 4.6, C is not a theorem of S5.

(b) $\Longrightarrow$ (a) Let us assume that $P \vee R$ is a theorem in the propositional calculus, then

(i)	$\vdash P \vee R$	[1.5]
(ii)	$\vdash R \to (\Diamond R)$	[1.8.(16)]
(iii)	$\vdash P \vee (\Diamond R)$	[1.7, (i) and (ii)]
(iv)	$\vdash P \vee (\Box Q_1) \vee \cdots \vee (\Box Q_k) \vee (\Diamond R)$	[1.7 and (iii)]

Let us assume, now, that there exists an i, $1 \leq 1 \leq k$, such that $Q_i \vee R$ is a theorem in the propositional calculus, then

(i) $\vdash Q_i \vee R$	[1.5]
(ii) $\vdash R \rightarrow (\Diamond R)$	[1.8.(15)]
(iii) $\vdash Q_i \vee (\Diamond R)$	[1.7, (i) and (ii)]
(iv) $\vdash \Box(Q_i \vee (\Diamond R))$	[necessarity rule and (iii)]
(v) $\vdash (\Box Q_i) \vee (\Diamond R)$	[modus ponens, 1.8.(15) and (iv)]
(vi) $\vdash P \vee (\Box Q_1) \vee \cdots \vee (\Box Q_k) \vee (\Diamond R)$	[1.7 and (v)]

∎

6.- Implementation and examples

In this section we give an algorithm for S5. Its correctness is an immediate consequence of the preceedings results. It is directly implementable in Lisp.

Algorithm 6.1. (*proof*)

Input: A formula P of S5.

Output: "yes", if $\vdash P$; "not", otherwise.

Procedure: *proof*(P)

begin
  $Q := f_3(f_5(f_2(f_1(P))))$
  $C_1, \ldots, C_n$ are clauses such that $Q = C_1 \wedge \cdots \wedge C_n$
  $i := 1$
  while $i \leq n$ **do**
    begin
      if *proof-clause*(C_i) = "yes" **then** $i := i + 1$
        else return "not"
    end
  return "yes"
end

Algorithm 6.2. (*proof-clause*)

Input: A clause $C = P_1 \vee \cdots \vee P_n$.

Output: "yes", if $\vdash C$; "not", otherwise.

Procedure: proof-clause(C)

```
begin
  {P_1, ..., P_s} := {P ∈ {P_1, ..., P_n} : P is a literal}
  P := P_1 ∨ ··· ∨ P_s
  {R_1, ..., R_u} := {R : ◇R ∈ {P_1, ..., P_n}}
  R := R_1 ∨ ··· ∨ R_u
  if tautology(P ∨ R) = "yes" then return "yes"
    else {Q_1, ..., Q_t} := {Q : □Q ∈ {P_1, ..., P_n}}
         i := 1
         while i ≤ t do
           begin
             if tautology(R ∨ Q_i) = "yes" then return "yes"
               else i := i + 1
           end
         return "not"
end
```

Algorithm 6.3. (*tautology*)

Input: A formula P of $\mathbf{P}_2$ with degree 0.

Output: "yes", if $\vdash P$; "not", otherwise.

Procedure: tautology(P)

```
begin
  Q := f_3(P)
  C_1, ..., C_n are basic clauses such that Q := C_1 ∧ ··· ∧ C_n
  i := 1
  while i ≤ n do
    begin
      if basic-proof-clause(C_i) = "yes" then i := i + 1
        else return "not"
    end
  return "yes"
end
```

Algorithm 6.4. (*basic-proof-clause*)

Input: A basic clause $C = P_1 \vee \cdots \vee P_n$.

Output: "yes", if $\vdash C$; "not", otherwise.

Procedure: basic-proof-clause(C)

```
begin
  LP := {Q ∈ {P_1, ..., P_n} : Q is a propositional variable}
  LN := {R : ¬R ∈ {P_1, ..., P_n}}
  if LP ∩ LN = ∅ then return "not"
    else return "yes"
end
```

Example 6.5. We are going to apply the preceedings algorithms to the following formula:

$$\Box\Box(p \to (q \wedge \Diamond r)) \to \neg\Diamond(p \wedge \neg q \wedge \neg\Diamond r)$$

Using 6.1 we compute

$$\mathit{proof}(\Box\Box(p \to (q \wedge \Diamond r)) \to \neg\Diamond(p \wedge \neg q \wedge \neg\Diamond r))$$

$$\begin{aligned}
& f_3(f_5(f_2(f_1(\Box\Box(p \to (q \wedge \Diamond r)) \to \neg\Diamond(p \wedge \neg q \wedge \neg\Diamond r))))) = \\
& = f_3(f_5(f_2(\neg\Box\Box(\neg p \vee (q \wedge \Diamond r)) \vee \neg\Diamond(p \wedge \neg q \wedge \neg\Diamond r)))) = \\
& = f_3(f_5(\Diamond\Diamond(p \wedge (\neg q \vee \Box\neg r)) \vee \Box(\neg p \vee q \vee \Diamond r))) = \\
& = f_3(f_5(\Diamond\Diamond(p \wedge (\neg q \vee \Box\neg r))) \vee f_5(\Box(\neg p \vee q \vee \Diamond r))) = \\
& \qquad\qquad [\text{by } 3.3.(2.1)] \\
& = f_3(f_5(\Diamond(p \wedge (\neg q \vee \Box\neg r))) \vee f_5(\Diamond r) \vee \Box(\neg p \vee q)) = \\
& \qquad\qquad [\text{by } 3.3.(2.8 \text{ y } 2.4.1)] \\
& = f_3(f_5(\Diamond(f_4(p \wedge (\neg q \vee \Box\neg r))) \vee \Diamond r \vee \Box(\neg p \vee q)) = \\
& \qquad\qquad [\text{by } 3.3.(2.7.2 \text{ y } 1)] \\
& = f_3(f_5(\Diamond((p \wedge \neg q) \vee (p \wedge \Box\neg r))) \vee \Diamond r \vee \Box(\neg p \vee q)) = \\
& = f_3(f_5(\Diamond(p \wedge \neg q)) \vee f_5(\Diamond(p \wedge \Box\neg r)) \vee \Diamond r \vee \Box(\neg p \vee q)) = \\
& \qquad\qquad [\text{by } 3.3.(2.6)] \\
& = f_3(\Diamond(p \wedge \neg q) \vee (\Diamond p \wedge f_5(\Box\neg r)) \vee \Diamond r \vee \Box(\neg p \vee q)) = \\
& \qquad\qquad [\text{by } 3.3.(1 \text{ y } 2.7.1)] \\
& = f_3(\Diamond(p \wedge \neg q) \vee (\Diamond p \wedge \Box\neg r) \vee \Diamond r \vee \Box(\neg p \vee q)) = \\
& \qquad\qquad [\text{by } 3.3.(1)] \\
& = (\Diamond(p \wedge \neg q) \vee \Diamond p \vee \Diamond r \vee \Box(\neg p \vee q)) \wedge (\Diamond(p \wedge \neg q) \vee \Box\neg r \vee \Diamond r \vee \Box(\neg p \vee q))
\end{aligned}$$

Using 6.2 we compute

$$\mathit{proof\text{-}clause}(\Diamond(p \wedge \neg q) \vee \Diamond p \vee \Diamond r \vee \Box(\neg p \vee q))$$

$\{R_1, R_2, R_3\} := \{p \wedge \neg q, p, r\}$
$R := (p \wedge \neg q) \vee p \vee r$
$\mathit{tautology}((p \wedge \neg q) \vee p \vee r) =$ "not"
$\{Q_1\} = \{\neg p \vee q\}$
$\mathit{tautology}((p \wedge \neg q) \vee p \vee r \vee \neg p \vee q) =$ "yes"

Then,

$$\mathit{proof\text{-}clause}(\Diamond(p \wedge \neg q) \vee \Diamond p \vee \Diamond r \vee \Box(\neg p \vee q)) = \text{“yes”}$$

Using 6.2 we compute

$$\mathit{proof\text{-}clause}(\Diamond(p \wedge \neg q) \vee \Box\neg r \vee \Diamond r \vee \Box(\neg p \vee q))$$

$\{R_1, R_2\} := \{p \wedge \neg q, r\}$
$R := (p \wedge \neg q) \vee r$
$\mathit{tautology}((p \wedge \neg q) \vee r) =$ "not"

$\{Q_1, Q_2\} := \{\neg r, \neg p \vee q\}$
$tautology((p \wedge \neg q) \vee r \vee \neg r) =$ "yes"
$tautology((p \wedge \neg q) \vee r \vee \neg p \vee q) =$ "yes"
Then,

$$proof\text{-}clause(\Diamond(p \wedge \neg q) \vee \Box\neg r \vee \Diamond r \vee \Box(\neg p \vee q)) = \text{"yes"}$$

Therefore,

$$proof(\Box\Box(p \rightarrow (q \wedge \Diamond r)) \rightarrow \neg\Diamond(p \wedge \neg q \wedge \neg\Diamond r)) = \text{"yes"}$$

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