# On $\Sigma_n$–definability in Arithmetic

Borrego-Díaz, J.; Fernández–Margarit, A.; Pérez-Jiménez; M.J. *

Computer Science and Artificial Intelligence
University of Seville (SPAIN)
E-mail:marper@cica.es

**Abstract** The proper subsets, $X$, of an $\mathcal{L}$–structure $M$ whose elements are the only ones that are $\Sigma_n$–definable with parameters in $X$ ("maximal $\Sigma_n$–definable set") are studied in this paper. Nonstandard sets are obtained which satisfies this condition. We connect them with the well know structures $K_n(M;X)$, $I_n(M;X)$. J. Paris and L. Kirby ([6]) gave a model of $\mathbf{B}\Sigma_{n+2} + \neg\mathbf{I}\Sigma_{n+2}$ that not satisfies this maximality condition. Now, another model of this theory will be given that verifies this property.

## 1 Introduction

Under certain assumptions, the classical structures $K_n(M;X)$ and $I_n(M;X)$ are models of induction but not collection, and collection but not induction, respectively, at the same level in the hierarchy of Arithmetic's fragments given by Paris–Kirby's diagram.

$$\begin{array}{ccccccc}
\mathbf{I}\Sigma_n & \Longleftrightarrow & \mathbf{I}\Pi_n & \Longleftrightarrow & \mathbf{L}\Pi_n & \Longleftrightarrow & \mathbf{L}\Sigma_n \\
\Uparrow & & & & & & \\
\mathbf{B}\Sigma_{n+1} & \Longleftrightarrow & \mathbf{B}\Pi_n & & & & \\
\Uparrow & & & & & & \\
\mathbf{I}\Sigma_{n+1} & & & & & &
\end{array}$$

Furthermore, in general,

(a) $K_n(M;X)$ is not an initial segment of $M$, but their elements are the only ones $\Sigma_n$–definables with parameters in $K_n(M;X)$.

(b) $I_n(M;X)$ is an initial segment of $M$, but there exists $\Sigma_n$–definable elements of $M - I_n(M;X)$ with parameters in $I_n(M;X)$.

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The aim of this paper is

- To describe new structures, $J_\Gamma(M;X)$, that for certain models, $M$, of a fragment of Arithmetic verify richer properties than the above mentioned for classical structures $K_n(M;X)$ and $I_n(M;X)$.
- To give a functional description of Hájek–Pudlák's structures $H^n(M;X)$, through $J_{\Sigma_n}(M;X)$.

Finally, we conclude by proposing some open questions.

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