On Spiking Neural P systems

August 29, 2008

Definition 0.1 *Given a P system,* Π*, the computation tree,* $T(\Pi)$*, associated with* Π *is the rooted labelled tree defined as follows:*

- *The nodes of the tree are labelled by configurations of* Π.
- *The label of the root is the initial configuration of* Π.
- *The children of a node are labelled by the configurations obtained from the configuration labelling the node through a step of transition.*

The maximal branches of the computation tree associated with a P system are called *computations* of the system.

It is possible to consider a orientation in the computation tree, $T(\Pi)$, in a natural way through the parent–child relation, that is, (u, v) is an oriented arc in the tree if and only if u is the parent of v (or v is a child of u).

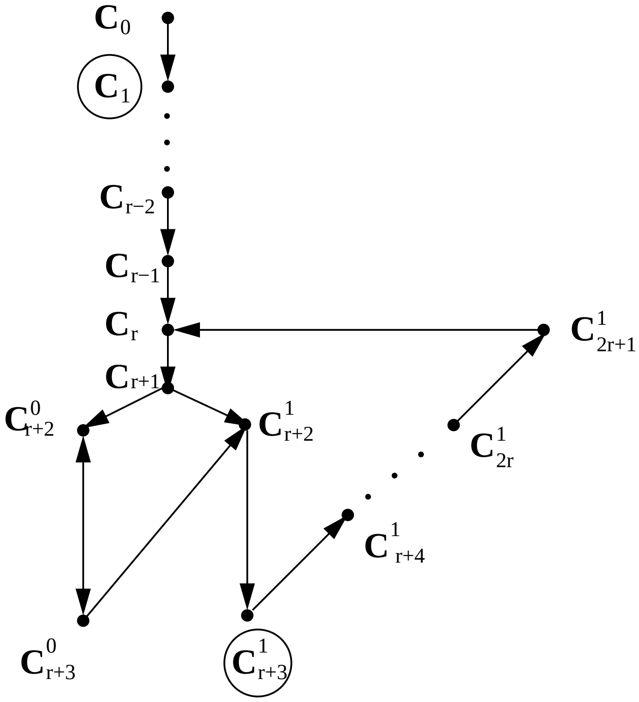
Definition 0.2 *Given a neural P system,* Π*, the oriented graph associated with* Π *is obtained from the (oriented) computation tree of* Π*, by identifying the nodes having the same label.*

1 About the SN P system Π from Figure 8

A configuration, C, of Π can be described by a $(6+r)$–tuple $C = (C(1), \dots, C(6+r))$ where $C(j)$ is the multiset over $\{a\}$ contained in neuron j, for $1 \le j \le 6$, in neuron c_{j-6}, for $7 \le j \le 5+r$, or in neuron c'_{r-1} for $j = 6+r$.

For each configuration C we denote by C^δ (with $\delta = 0, 1$) the configuration obtained from C applying in neuron 2 the rule $a \to a; \delta$.

It is easy to prove that the only configurations sending a spike to the neighbouring neurons are C_1 and C^1_{r+3}, and that the following equalities hold $C^{00}_{r+4} = C^0_{r+2}$ and $C^{01}_{r+4} = C^1_{r+2}$. Hence, the oriented graph associated with the SN P system Π can be depicted as follows:



From the oriented graph associated with Π it can be observed that there is no halting computation.

We denote by σ the path $C_0 \to C_1 \to \ldots \to C_r \to C_{r+1}$ and we denote by τ the path $C^1_{r+3} \to C^1_{r+4} \to \ldots \to C^1_{2r+1} \to C_r \to C_{r+1}$. Then, Length$(\sigma) = r+1$ and Length$(\tau) = r$.

For each $j \geq 0$, we denote by $\gamma(j)$ the path

$C_{r+1} \to C^0_{r+2} \overset{(j)}{\longleftrightarrow} C^0_{r+3} \to C^1_{r+2} \to C^1_{r+3}$

meaning that the computation goes through C^0_{r+2} exactly j times (for example, $\gamma(0)$ is the path $C_{r+1} \to C^1_{r+2} \to C^1_{r+3}$, $\gamma(1)$ is the path $C_{r+1} \to C^0_{r+2} \to C^0_{r+3} \to C^1_{r+2} \to C^1_{r+3}$, and $\gamma(2)$ is the path $C_{r+1} \to C^0_{r+2} \to C^0_{r+3} \to C^0_{r+2} \to C^0_{r+3} \to C^1_{r+2} \to C^1_{r+3}$). Then, Length$(\gamma(j)) = 2(j+1)$.

Taking into account that the computations of Π are the maximal branches of the computation tree, for every computation, $\mathcal{C}$, of Π there exists an infinite sequence of natural numbers $\{i_k : \ k \geq 1\}$ such that the computation $\mathcal{C}$ can be described through the following path in the graph associated with Π:

$\sigma\gamma(i_1)\tau\gamma(i_2)\tau\gamma(i_3)\tau\gamma(i_4)\tau\gamma(i_5)\ldots\ldots$

We will denote that computation by $\mathcal{C}(\{i_k : \ k \geq 1\})$.

Now, we describe the spike train of computation $\mathcal{C}(\{i_k : \ k \geq 1\}) \equiv C_0 \Rightarrow C_1 \Rightarrow C_2 \Rightarrow C_3 \ldots$, computing the sequence of steps i such that the configuration C_i sends a spike out. We have:

$t_1 = 1;\ t_2 = t_1 + r + 2(i_1+1);\ t_3 = t_2 + r + 2(i_1+1);\ t_4 = t_3 + r + 2(i_1+1); \ldots$

That is

$$\begin{cases} t_1 & = & 1 \\ t_{k+1} & = & t_k + r + 2(i_k+1) \end{cases}$$

Hence, $N_{all}(\mathcal{C}(\{i_k : \ k \geq 1\}) = \{t_{k+1} - t_k : \ k \geq 1\} = \{r + 2(i_k+1) : \ k \geq 1\}$.

Theorem 1.1

(*a*) *For each computation,* $\mathcal{C}$ *of* Π *we have* $N(\mathcal{C}) \subseteq \{r + 2i :\ i \geq 1\}$ *(correctness).*

(*b*) *For each* $i \geq 1$ *there exists a computation,* $\mathcal{C}$ *of* Π *such that* $r + 2i \in N(\mathcal{C})$ *(completeness). Moreover, The SN P system* Π *is weakly* ω*–coherent.*

Proof.

(*a*) Let $\mathcal{C}$ be a computation of Π. Then there exists an infinite sequence of natural numbers $\{i_k :\ k \geq 1\}$ such that $\mathcal{C} = \mathcal{C}(\{i_k :\ k \geq 1\})$. Then, $N(\mathcal{C}(\{i_k :\ k \geq 1\}) = \{r+2(i_k+1) :\ k \geq 1\} \subseteq \{r+2i :\ i \geq 1\}$. Hence $N_{all}(\Pi) \subseteq \{r+2i :\ i \geq 1\}$

(*b*) Let us prove that there exists a computation $\mathcal{C}$ of Π such that $N(\mathcal{C}) = N_{all}(\Pi) = \{r + 2i :\ i \geq 1\}$.

Indeed, let s be the infinite sequence $\{i_k :\ k \geq 1\}$ such that $i_k = k - 1$, for each $k \geq 1$. Then, $N(\mathcal{C}(s)) = \{r + 2(i_k + 1) :\ k \geq 1\} = \{r + 2i :\ i \geq 1\}$. Hence $\{r+2i :\ i \geq 1\} = \{r+2(i_k+1) :\ k \geq 1\} = N(\mathcal{C}(s)) \subseteq N_{all}(\Pi) \subseteq \{r+2i :\ i \geq 1\}$.

□

Proposition 1.1 *For each* $q \geq 2$ *we denote by* $N_{q^*}(\Pi)$ *the set*
$\{t_q - t_{q-1}|\ st(\mathcal{C}) = \langle t_1, \ldots, t_{q-1}, t_q, \ldots\rangle,$ *for some computation* $\mathcal{C}$ *of* $\Pi\}$.
Then we have $N_{q^*}(\Pi) = N_{all}(\Pi) = \{r + 2i :\ i \geq 1\}$.

Proof. If $\{i_k :\ k \geq 1\}$ is an infinite sequence of natural numbers, then

$$N(\mathcal{C}(\{i_k :\ k \geq 1\})) = \{r + 2(i_k + 1) :\ k \geq 1\}$$

For each $j \geq 0$ let $\mathcal{C}_j = \mathcal{C}(\{i_k :\ k \geq 1\}$, where $i_k = j$, for every $k \geq 1$.

Let $q \geq 2$. For each $j \geq 0$ we have $N_{q^*}(\mathcal{C}_j) = \{t_q - t_{q-1}\} = \{r + 2(j + 1)\}$. On one hand,

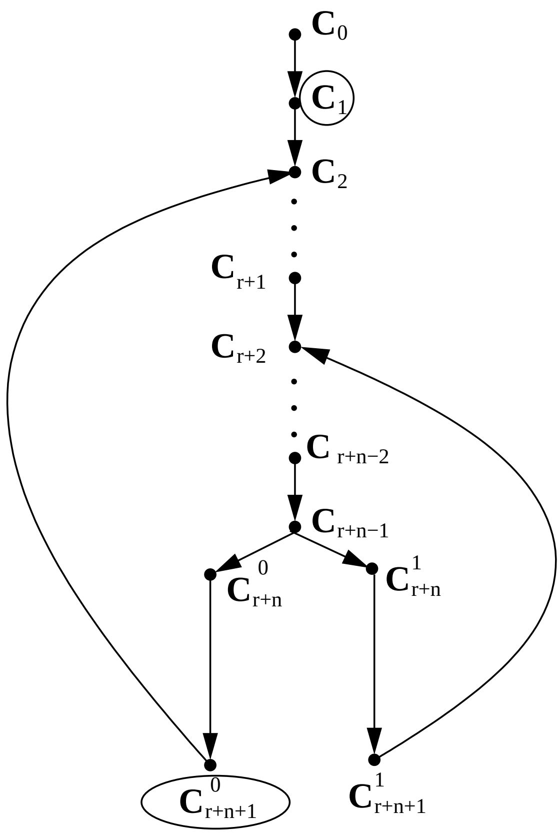
$\{r + 2i :\ i \geq 1\} = \{r + 2(j + 1) :\ j \geq 0\} = \{N_{q^*}(\mathcal{C}_j) :\ j \geq 0\} \subseteq N_{q^*}(\Pi)$.
On the other hand, $N_{q^*}(\Pi) \subseteq N_{all}(\Pi) = \{r + 2i :\ i \geq 1\}$.
Hence, $N_{q^*}(\Pi) = \{r + 2i :\ i \geq 1\}$. □

2 About the SN P system Π from Figure 9

A configuration, C, of Π can be described by a $(n+r+2)$–tuple $C = (C(0), \ldots, C(n+r+1))$ where $C(j)$ is the multiset over $\{a\}$ contained in neuron j, for $0 \leq j \leq n$, in neuron d_{j-n-1}, for $n + 2 \leq j \leq n + r$, or in neuron *out* for $j = n + r + 1$.

For each configuration C we denote by C^δ (with $\delta = 0, 1$) the configuration obtained from C applying in neuron 0 the rule $a \to a; \delta$.

It is easy to prove that the only configurations sending a spike to the neighbouring neurons are C_1 and C^0_{r+n+1}, and that the following equalities hold $C_2 = C^0_{r+n+2}$ and $C^1_{r+n+2} = C_{r+2}$. Hence, the oriented graph associated with the SN P system Π can be depicted as follows:



Analysing the oriented graph associated with Π we note that there are not any halting computation.

We denote by σ the path $C_2 \rightarrow \ldots \rightarrow C_{r+2}$ and we denote by τ the path $C_{r+2} \rightarrow \ldots C_{r+n-1} \rightarrow C^0_{r+n} \rightarrow C^0_{r+n+1}$. Then, Length($\sigma$) $= r$ and Length(τ) $= n - 1$.

For each $i \geq 0$, we denote by $\gamma(i)$ the path

$C_{r+2} \xrightarrow{(i)} \ldots \rightarrow C_{r+n-1} \rightarrow C^1_{r+n} \rightarrow C^1_{r+n+1} \rightarrow C^1_{r+2}$

meaning that the configuration C_{r+2} is passed exactly $i+1$ times (for example, the path $\gamma(0)$ contain only the node C_{r+2}, the path $\gamma(1)$ is $C_{r+2} \rightarrow \ldots \rightarrow C_{r+n-1} \rightarrow C^1_{r+n} \rightarrow C^1_{r+n+1} \rightarrow C^1_{r+2}$). Then, Length($\gamma(i)$) $= ni$.

Let $\delta(i)$ the following path $C_2 \overset{\sigma}{\rightsquigarrow} C_{r+2} \overset{\gamma(i)}{\rightsquigarrow} C_{r+2}$. That is, through the path $\delta(i)$ we go from node C_2 to node C_{r+2} without going through node C^0_{r+2} but passing $(i+1)$ times by node C_{r+2}. Then, Length($\delta(i)$) $= r + ni$.

Taking into account that the computations of Π are the maximal branches of the computation tree, for every computation, $\mathcal{C}$, of Π there exists an infinite sequence of natural numbers $\{i_k : \ k \geq 1\}$ such that the computation $\mathcal{C}$ can be described through the following path in the graph associated with Π:

$C_0 \to C_1 \to C_2 \overset{\delta(i_1)}{\rightsquigarrow} C_{r+2} \overset{\tau}{\rightsquigarrow} C^0_{r+n+1} \to C_2 \overset{\delta(i_2)}{\rightsquigarrow} C_{r+2} \overset{\tau}{\rightsquigarrow} C^0_{r+n+1} \to \dots$
We will denote that computation by $\mathcal{C}(\{i_k : \ k \geq 1\})$.

Now, we describe the spike train of computation $\mathcal{C}(\{i_k : \ k \geq 1\}) \equiv C_0 \Rightarrow C_1 \Rightarrow C_2 \Rightarrow C_3 \dots$, computing the sequence of steps i such that the configuration C_i sends a spike out. We have:

$t_1 = 1; \ t_2 = t_1 + r + n(i_1 + 1); \ t_3 = t_2 + r + n(i_2 + 1); \ t_4 = t_3 + r + n(i_1 + 1); \dots$
That is

$$\begin{cases} t_1 & = & 1 \\ t_{k+1} & = & t_k + r + n(i_k + 1) \end{cases}$$

Hence, $N_{all}(\mathcal{C}(\{i_k : \ k \geq 1\}) = \{t_{k+1} - t_k : \ k \geq 1\} = \{r + n(i_k + 1) : \ k \geq 1\}$.

Theorem 2.1 *Let $n \geq 2$.*

- (a) *For each computation, $\mathcal{C}$ of Π we have $N(\mathcal{C}) \subseteq \{r + ni : \ i \geq 1\}$ (correctness).*
- (b) *For each $i \geq 1$ there exists a computation, $\mathcal{C}$ of Π such that $r + ni \in N(\mathcal{C})$ (completeness). Moreover, The SN P system Π is weakly ω–coherent.*
- (c) *$N_{q^*}(\Pi) = N_{all}(\Pi) = \{r + ni : \ i \geq 1\}$.*

Proof. Similar to proof of Theorem 1.1 and Proposition 1.1 changing $r + 2i$ by $r + ni$. □

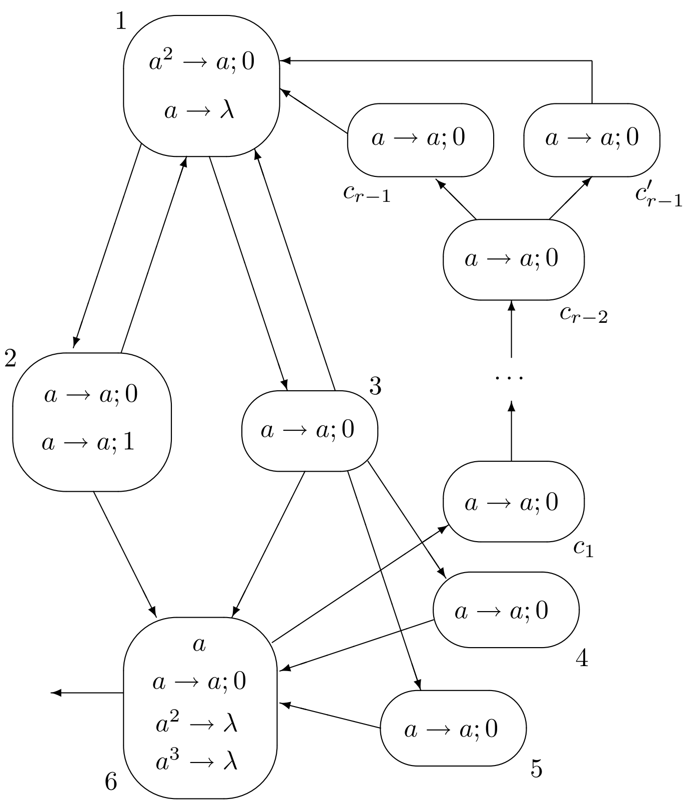


Fig. 8. An SN P system computing $\{r + 2i \mid i \geq 1\}$

Lemma 8.4 (for each SN P system Π there is an equivalent SN P system Π' containing initially only one spike inside one of its neurons) and Lemma 8.5 (all families of numbers computed by systems with at least two rules, two consumed or used spikes, as well as with at least two spikes contained in the neurons, are closed under union) from [4] are true also for our cases (with the same proofs as in [4]).

Using this "union lemma", we ensure that all semilinear sets belong to families of the form $Spik_\alpha^\beta P_*(rule_k, cons_p, forg_q, bound_*)$ with small values of parameters k, p, q, but not for the case when we consider computations which halt after spiking. For instance, Lemma 7 ensures only that the system does not send out more than k spikes, but the computation can continue forever. However, this can be fixed in the case of bounded computations for any initial SN P system, not only for a particular one, as was the case in the proof of Theorem 6:

Lemma 12. *Given a system Π and a threshold s on the number of spikes in any neuron, we can construct a system Π_k such that:*

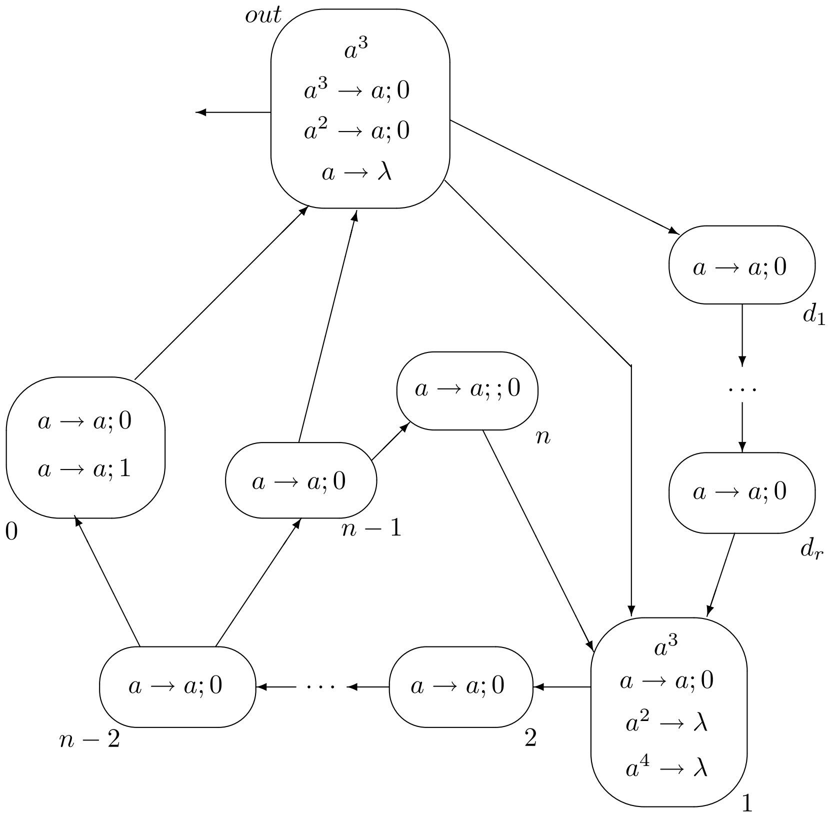


Fig. 9. An SN P system computing $\{r + ni \mid i \geq 1\}$, for $n \geq 3$

(1) for each s-bounded computation γ of Π with $st(\gamma) = \langle t_1, t_2, \ldots, t_j \rangle$, $j \leq k$, there is a halting 2s-bounded computation γ' of Π' such that $st(\gamma') = \langle t_1 + 2, t_2 + 2, \ldots, t_j + 2 \rangle$;

(2) for each s-bounded computation γ of Π with $st(\gamma) = \langle t_1, t_2, \ldots, t_k, \ldots \rangle$, there is a halting 2s-bounded computation γ' of Π' such that $st(\gamma') = \langle t_1 + 2, t_2 + 2, \ldots, t_k + 2 \rangle$;

(3) each computation γ' of Π' either (i) never spikes, or (ii) $st(\gamma') = \langle t_1 + 2, t_2 + 2, \ldots, t_j + 2 \rangle$ for some computation γ in Π with $st(\gamma) = \langle t_1, t_2, \ldots, t_j \rangle$, $j \leq k$, or (iii) $st(\gamma') = \langle t_1 + 2, t_2 + 2, \ldots, t_k + 2 \rangle$ for some computation γ in Π with $st(\gamma) = \langle t_1, t_2, \ldots, t_k, \ldots \rangle$.

Proof. For a given $k \geq 1$ and an SN P system Π we construct the system Π' as follows (without loss of generality, we may assume that all rules of Π are of the form $a^j \to x$, for some $j \leq s$, because only such rules can be useful in computations –